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ANALYTIC HAUSDORFF GAPS II: THE DENSITY ZERO IDEAL

BY

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ABSTRACT

We prove two results about the quotient over the asymptotic density zero ideal. First, it is forcing equivalent to $\mathcal{P}(\mathbb{N})/\operatorname{Fin} * \mathcal{R}_{\mathfrak{c}}$, where $\mathcal{R}_{\mathfrak{c}}$ is the homogeneous probability measure algebra of character $\mathfrak{c}$. Second, if it has analytic Hausdorff gaps, then they look considerably different from previously known gaps of this form.

Introduction

We consider **density ideals**, ideals of the form $\mathcal{Z}_\mu = \{A \mid \limsup_n \mu_n(A) = 0\}$ for a sequence μ_m ($m \in \mathbb{N}$) of probability measures concentrating on pairwise disjoint intervals I_m ($m \in \mathbb{N}$) of $\mathbb{N}$. In Theorem 1.3 we prove that the regular open algebra of such quotient is isomorphic to the regular open algebra of $\mathcal{P}(\mathbb{N})/\operatorname{Fin} * \mathcal{R}_{\mathfrak{c}}$. Study of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ as forcing notions has recently attracted a bit of attention ([1], [12], [8]).

In [19] it was proved that there are no analytic Hausdorff gaps over Fin. Todorcevic actually proved that every pregap $\mathcal{A}$, $\mathcal{B}$ over Fin such that $\mathcal{A}$ is analytic and $\mathcal{B}/\operatorname{Fin}$ is σ-directed can be countably separated (and more). In [3, Theorem 5.7.1, Theorem 5.7.2 and Lemma 5.8.7] we have proved that Fin is the

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only analytic P-ideal that has this property: If $\mathcal{I}$ is an analytic P-ideal that is not Rudin–Keisler isomorphic to Fin, then there is a gap $\mathcal{A}$, $\mathcal{B}$ over $\mathcal{I}$ such that $\mathcal{A}$ and $\mathcal{B}$ are Borel, $\mathcal{B}/\mathcal{I}$ is σ-directed and $\mathcal{A}$ is not countably separated from $\mathcal{B}$.

In [4] it was proved that there are analytic Hausdorff gaps over any dense F_σ P-ideal. Recall that

$$\mathcal{Z}_0 = \{A \subseteq \mathbb{N} : \limsup_n |A \cap n|/n = 0\}$$

is the ideal of *asymptotic density zero* sets. In §2 we prove results on the structure of analytic Hausdorff gaps in its quotients, making some progress towards [3, Question 5.13.7] and [4, Question 8a and Question 10].

In Proposition 3.2 we show that if $\mathcal{I}$ is a dense analytic P-ideal without analytic Hausdorff gaps in its quotient, then the restriction of $\mathcal{I}$ to some positive set is a generalized density ideal. This gives a partial solution to the problem of characterizing those analytic P-ideals that do not have analytic Hausdorff gaps in their quotients ([3, Problem 5.13.5]; see also Question 4.1).

TERMINOLOGY. Our terminology and notation follow [3]. Two families $\mathcal{A}, \mathcal{B}$ in a quotient $\mathcal{P}(\mathbb{N})/\mathcal{I}$ form a **pregap** if $A \cap B \in \mathcal{I}$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. A pregap is **separated** (or **split**) by $C \subseteq \mathbb{N}$ if for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \setminus C \in \mathcal{I}$ and $B \cap C \in \mathcal{I}$. If a pregap is not separated by any C, then it is a **gap**. We also say that $\mathcal{A}$ and $\mathcal{B}$ form a gap **over** $\mathcal{I}$. A pregap is **countably separated** if there are sets $C_n \subseteq \mathbb{N}$ $(n \in \mathbb{N})$ such that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ there is n such that $A \setminus C_n \in \mathcal{I}$ and $B \cap C_n \in \mathcal{I}$. A gap is **Hausdorff** if both of its sides $\mathcal{A}$ and $\mathcal{B}$ are countably directed under inclusion modulo $\mathcal{I}$. A gap is **analytic** if $\mathcal{A}$ and $\mathcal{B}$ are analytic subsets of $\mathcal{P}(\mathbb{N})$, taken with its Cantor-set topology.

An ideal $\mathcal{I}$ on $\mathbb{N}$ is a **P-ideal** if for every sequence A_n of sets in $\mathcal{I}$ there is $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n. An ideal $\mathcal{I}$ is **dense** if every infinite $A \subseteq \mathbb{N}$ has an infinite subset in $\mathcal{I}$.

A function ϕ defined on the power-set of some set I is a **submeasure** if $\phi(\emptyset) = 0$, it is monotonic ($A \subseteq B$ implies $\phi(A) \leq \phi(B)$), and subadditive ($\phi(A \cup B) \leq \phi(A) + \phi(B)$). We say that ϕ is a **submeasure on** I. A submeasure on $\mathcal{P}(\mathbb{N})$ is **lower semicontinuous** if for all A we have $\phi(A) = \sup \phi(s)$, where s ranges over all finite subsets of A. In this case

$$\mathrm{Exh}(\phi) = \{A \mid \limsup_n \phi(A \setminus n) = 0\}$$

is an analytic P-ideal, and all analytic P-ideals are of this form ([16]).

Fin is the ideal of all finite subsets of $\mathbb{N}$. Ideals I and J are **Rubin–Keisler isomorphic** if there are $A \in I$, $B \in J$, and a bijection $h: \mathbb{N} \setminus A \to \mathbb{N} \setminus B$ such that $h^{-1}(C) \in I$ if and only if $C \in J$ for all $C \in \mathbb{N} \setminus B$.

If $\mathbb{N} = \bigcup_n I_n$ is a partition into finite intervals and ϕ_n is a submeasure on I_n, then

$$\mathcal{Z}_\phi = \{A \subseteq \mathbb{N} | \limsup_n \phi_n(A \cap I_n) = 0\}$$

is a typical **generalized density ideal** (see [3, §13]). These ideals are $F_{\sigma\delta}$ subsets of $\mathcal{P}(\mathbb{N})$ (when taken in its natural Cantor-set topology). Each $\mathcal{Z}_\phi$ is a P-ideal, and it is dense if and only if $\limsup_i \sup_n \phi_i(\{n\}) = 0$.

If each ϕ_n is a measure ν_n, then $\mathcal{Z}_\nu$ is a **density ideal**. It is an **EU-ideal** if it is dense and $\nu_n(I_n) = 1$ for all n. This is not the original definition given in [14], but in [3, Theorem 1.13.3 (b)] the two conditions were proved to be equivalent. By [3, p. 48] $\mathcal{Z}_0$ is an EU-ideal and a density ideal $\mathcal{Z}_\nu$ is an EU-ideal if and only if $\sup_n \nu_n(I_n) < \infty$.

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1. Ultraproducts of measure algebras

By $[C]^\infty$ we denote the family of all infinite subsets of C. In this section C will always stand for an infinite subset of $\mathbb{N}$. For $n \in \mathbb{N}$ let $C/n = C \setminus (n+1)$. A family $\mathcal{F} \subseteq [\mathbb{N}]^\infty$ is **dense** if for every $C \in [\mathbb{N}]^\infty$ we have $\mathcal{F} \cap [C]^\infty \neq \emptyset$. An ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is **selective** if it intersects every dense analytic subset of $[\mathbb{N}]^\infty$. By the localized version of Silver's theorem due to Mathias ([15]) this is equivalent to the standard definition of a selective ultrafilter.

Lemma 1.2 is well-known. The use of a selective ultrafilter in the context of Loeb measure dates back to [11] and it was studied in [2].

LEMMA 1.1: *Assume $\mathbb{A}_n$ is a finite Boolean algebra with submeasure ϕ_n and $\mathcal{U}$ is a selective ultrafilter. On the ultraproduct $\mathbb{A} = (\prod_n \mathbb{A}_n)/\mathcal{U}$ define*

$$\phi_{\mathcal{U}}([V]_{\mathcal{U}}) = \lim_{m \to \mathcal{U}} \phi_m(V_m).$$

Then $\phi_{\mathcal{U}}$ is a countably subadditive submeasure.

Proof: Clearly $\phi_{\mathcal{U}}$ is a well-defined submeasure on $\mathbb{A}$. We first prove $\phi_{\mathcal{U}}$ is countably subadditive. Pick $B^n \in \mathbb{A}$ ($n \in \mathbb{N}$) so that $\phi_{\mathcal{U}}(B^n \cap B^m) = 0$ for $m \neq n$. Write $B^n = (B_i^n)/\mathcal{U}$ (where $B_i^n \in \mathbb{A}_i$). The families

$$\mathcal{D} = \left\{ C : (\forall i \in C)(\forall j \in C/i) \left| \sum_{n<i} \nu_j(B_j^n) - \nu_j\Big(\bigcup_{n<i} B_j^n\Big) \right| < 2^{-2i} \right\}$$

and

$$\mathcal{F}_n = \left\{ C : (\forall i \in C) |\nu_i(B_i^n) - \nu_{\mathcal{U}}(B^n)| < 2^{-i} n^{-1} \right\}$$

are dense in $[\mathbb{N}]^\infty$. Since $\mathcal{U}$ is selective, we can pick $C \in \mathcal{U} \cap \mathcal{D}$ such that $C/n \in \mathcal{F}_n$ for all n. Let m_i ($i \in \mathbb{N}$) be an increasing enumeration of C. Define $B = [B_k]_{\mathcal{U}}$ by $B_{m_{i+1}} = \bigcup_{n<m_i} B^n_{m_{i+1}}$ and $B_k = \emptyset$ for $k \notin C$. Then $B \supseteq B^n$ for all n, and for all pairs $i < j$ in C we have

$$\left| \phi_j(B_j) - \phi_j\Big(\bigcup_{n<i} B^n\Big) \right| < \left| \phi_j(B_j) - \phi_j\Big(\bigcup_{n<i} B_j^n\Big) \right| + 2^{-i} < 2^{-i+1},$$

hence $\phi_{\mathcal{U}}(B) = \lim_{i \to \mathcal{U}} \phi_i(B_i) = \lim_n \phi_{\mathcal{U}}(\bigcup_{i=1}^n (B^n))$. ∎

The finiteness of algebras $\mathbb{A}_n$ can obviously be replaced by the appropriate completeness assumption. It is not difficult to see that the algebra $\mathbb{A}/\operatorname{Null}(\phi_{\mathcal{U}})$ does not have to be σ-complete in general.

LEMMA 1.2: *Assume $(\mathbb{A}_n, \nu_n)$ are probability measure algebras and $\mathcal{U}$ is a selective ultrafilter. Then $\nu_{\mathcal{U}}$ is a countably additive probability measure and the quotient $\mathbb{A}/\operatorname{Null}(\nu_{\mathcal{U}})$ is a measure algebra.*

Proof: Clearly $\nu_{\mathcal{U}}$ is a finitely additive probability measure on $\mathbb{A}$, so $\mathbb{A}/\operatorname{Null}(\nu_{\mathcal{U}})$ is ccc. By Lemma 1.1, $\nu_{\mathcal{U}}$ is countably subadditive. Being in addition finitely additive, it is countably additive.

Let B^n and B be as in the proof of Lemma 1.1. In order to prove $\mathbb{A}/\operatorname{Null}(\nu_{\mathcal{U}})$ is σ-complete, it will suffice to check that B is the supremum of B_i. For $A \in \prod_{i=1}^\infty \mathbb{A}_n$ write $\bar{A} = [A]_{\operatorname{Null}(\nu_{\mathcal{U}})}$. We need to check $\bar{B} = \bigvee_n \bar{B}^n$ in $\mathbb{A}/\operatorname{Null}(\nu_{\mathcal{U}})$. Indeed, $\supseteq$ is immediate since $\bar{B} \supseteq \bar{B}^n$ for all n. To prove the reverse inclusion, note that $\bar{D} \subseteq \bar{B}$ and $\bar{D} \neq \bar{B}$ implies $\nu_{\mathcal{U}}(D) < \nu_{\mathcal{U}}(B)$. Then if m is large enough so that $\nu_{\mathcal{U}}(\bigcup_{n<m} B^n) > \nu_{\mathcal{U}}(D)$, we have $\bigcup_{n<m} \bar{B}^n \setminus \bar{D} \neq 0_{\mathbb{A}}$. Since D was arbitrary, this implies $\bar{B} = \bigvee_n \bar{B}^n$.

By ccc-ness, the algebra is complete and therefore a measure algebra. ∎

Let $\mathcal{R}_{\mathfrak{c}}$ denote the homogeneous probability measure algebra of Maharam character $\mathfrak{c}$ (see, e.g., [9]). The forcing terminology used in the proof of

Theorem 1.3 is standard. Neither forcing nor this theorem will be used elsewhere in this note.

THEOREM 1.3: *If $\mathcal{Z}_\nu$ is an EU-ideal, then the regular open algebras of $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$ and $\mathcal{P}(\mathbb{N})/\operatorname{Fin} * \mathcal{R}_\mathfrak{c}$ are isomorphic.*

First we prove a lemma. An ideal $\mathcal{I}$ is **proper** if $\mathbb{N} \notin \mathcal{I}$.

LEMMA 1.4: *If $\mathcal{Z}_\phi$ is a proper generalized density ideal, then $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ regularly embeds into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\phi$.*

Proof: The assumption that $\mathcal{Z}_\phi$ is proper is equivalent to $\limsup_n \phi_n(I_n) > 0$. We may assume $\liminf_n \phi_n(I_n) > 0$, by possibly joining some of the I_ns (see [3, §13]). Let h be a function that collapses I_n to n. We claim that $[A]_{\operatorname{Fin}} \mapsto [h^{-1}(A)]_{\mathcal{Z}_0}$ is a regular embedding. (Here $[A]_\mathcal{I}$ is the $\mathcal{I}$-equivalence class of $A \subseteq \mathbb{N}$.) It is clearly a homomorphism of Boolean algebras, and since $\liminf_n \phi_n(I_n) > 0$ it is also an embedding. Fix a maximal antichain $\mathcal{A}$ in $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$. We need to prove that $\{h^{-1}(A) | A \in \mathcal{A}\}$ is maximal over $\mathcal{Z}_\phi$. For $C \in \mathcal{Z}_\nu^+$ there is $\varepsilon > 0$ such that the set $\{n | \phi_n(C) > \varepsilon\}$ is infinite. By the maximality of $\mathcal{A}$, this set has an infinite intersection with some $A \in \mathcal{A}$, hence $h^{-1}(A) \cap C \notin \mathcal{Z}_\phi$. ∎

Proof of Theorem 1.3: We find a regular embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$ such that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ forces that the quotient is an atomless measure algebra. The character of this algebra is not bigger than its size, $\mathfrak{c}$. This suffices since $\mathcal{R}_\mathfrak{c}$ regularly embeds into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$ by [10, Proposition 491P]. Let $h\colon \mathbb{N} \to \mathbb{N}$ be a function that collapses I_n to n. By Lemma 1.4, the mapping $A \mapsto h^{-1}(A)$ is a regular embedding of $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ into $\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu$. Let G be the canonical name for some $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$-generic ultrafilter. Recall that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ adds no reals and forces that G is selective ([15]).

It remains to check that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ forces $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu)/G$ is isomorphic to $\mathcal{R}_\mathfrak{c}$. We will be using the terminology of Lemma 1.4. First prove that $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ forces $(\mathcal{P}(\mathbb{N})/\mathcal{Z}_\nu)/G$ and $(\prod_n \mathcal{P}(I_n)/G)/\operatorname{Null}(\nu_G)$ are isomorphic. Pick subsets B and C of $\mathbb{N}$. Identifying $\mathcal{P}(\mathbb{N})$ with $\prod_n \mathcal{P}(I_n)$, write $B_n = B \cap I_n$ and $C_n = C \cap I_n$. Then $B/G = C/G$ if and only if $\lim_{n\to G} \nu_n(B_n \Delta C_n) = 0$ if and only if $\nu_G([B]_G \Delta [C]_G) = 0$. Since G is forced to be a selective ultrafilter, by Lemma 1.4, the conclusion follows. ∎

Under CH it is even true that all quotients over EU-ideals are pairwise isomorphic ([14], [5]). However, under Todorcevic's OCA there are many pairwise nonisomorphic quotients over the ideals in this class (this was first proved by Just [13]; see also [3] and [7]).

2. Gaps over density ideals

In this section we prove a structure result on analytic Hausdorff gaps over density ideals. A pregap $\mathcal{A}, \mathcal{B}$ in the quotient over $\mathcal{Z}_\phi$ (given by I_n, ϕ_n for $n \in \mathbb{N}$) is **simple** if there are submeasures σ_m, τ_m concentrating on I_m such that $\mathcal{A} = \mathcal{Z}_\sigma$ and $\mathcal{B} = \mathcal{Z}_\tau$. If $\mathcal{A}, \mathcal{B}$ and $\mathcal{A}', \mathcal{B}'$ are pregaps in the same algebra, we say that $\mathcal{A}, \mathcal{B}$ is **included** in $\mathcal{A}', \mathcal{B}'$ if $\mathcal{A} \subseteq \mathcal{A}'$ and $\mathcal{B} \subseteq \mathcal{B}'$. Fix a generalized density ideal $\mathcal{Z}_\phi$ with witnesses ϕ_n and I_n, $n \in \mathbb{N}$, throughout this section.

THEOREM 2.1: *Every analytic Hausdorff pregap in the quotient over any $\mathcal{Z}_\phi$ is included in a simple pregap.*

Proof: Both $\mathcal{A}$ and $\mathcal{B}$ are analytic P-ideals. By [16], $\mathcal{A} = \mathrm{Exh}(\sigma)$ and $\mathcal{B} = \mathrm{Exh}(\tau)$ for some lower semicontinuous submeasures σ and τ. Define σ_m and τ_m by

$$\sigma_m(C) = \sigma(C \cap I_m) \qquad \text{and} \qquad \tau_m(C) = \tau(C \cap I_m),$$

and let $\mathcal{A}' = \mathcal{Z}_\sigma$, $\mathcal{B}' = \mathcal{Z}_\tau$.

CLAIM 2.2: *We have $\mathcal{A}' \supseteq \mathcal{A}$ and $\mathcal{B}' \supseteq \mathcal{B}$.*

Proof: For $\mathcal{A}' \supseteq \mathcal{A}$ it suffices to prove that $\sup_m \sigma_m \leq \sigma$. But this follows from $\sigma_m \leq \sigma$ for all m. The proof that $\mathcal{B}' \supseteq \mathcal{B}$ is analogous. ∎

CLAIM 2.3: *The families $\mathcal{A}'$ and $\mathcal{B}'$ are orthogonal over $\mathcal{Z}_\phi$.*

Proof: We need to check that $\mathcal{A}' \cap \mathcal{B}' \subseteq \mathcal{Z}_\phi$. Assume this fails, and fix $X \in (\mathcal{A}' \cap \mathcal{B}') \setminus \mathcal{Z}_\phi$. Let $X_m = X \cap I_m$. Since $X \notin \mathcal{Z}_\phi$, there is an $\varepsilon > 0$ such that $\phi_m(X_m) \geq \varepsilon$ for infinitely many m. We may assume this holds for all m. If we write

$$I_C = \bigcup_{m \in C} I_m, \quad X_C = X \cap I_C,$$

then for every infinite $C \subseteq \mathbb{N}$ we have $X_C \notin \mathcal{Z}_\phi$. Since $\mathcal{A} \cap \mathcal{B} \subseteq \mathcal{Z}_\phi$, we have $X_C \notin \mathcal{A} \cap \mathcal{B}$ for every such C. We may find an infinite C_0 and $Q \in \{\mathcal{A}, \mathcal{B}\}$ such that $\{D \in [C_0]^{\aleph_0} : X_D \notin Q\}$ is dense in $[C]^{\aleph_0}$ (dense in the forcing sense—every set has an infinite subset in this set). We may assume that $Q = \mathcal{A}$ and (since $\mathcal{A}$ is hereditary) that $X \cap I_D \notin \mathcal{A}$ for all $D \in [C_0]^{\aleph_0}$. For $D \subseteq \mathbb{N}$ let

$$\alpha_D = \liminf_k \sigma(X_D \setminus k)$$

and note that $\alpha_D > 0$ since $X_D \notin \mathcal{A}$. Since $C \subseteq^* D$ implies $\alpha_C \leq \alpha_D$ and $([\mathbb{N}]^{\aleph_0}, \supseteq^*)$ is countably directed, for some $C_1 \in [C_0]^{\aleph_0}$ we have $\alpha_D = \alpha_{C_1} = \delta$ for all $D \in [C_1]^{\aleph_0}$. By the above $\delta > 0$.

But $\sigma_m(X) = \sigma(X_m) \to 0$ as $m \to \infty$, so we can find $C_2 \subseteq C_1$ such that $\sum_{m \in C_2} \sigma(X_m) < \delta/2$. Then $\sigma(X_{C_2}) < \delta/2$, a contradiction. ∎

By the above claims, $\mathcal{A}'$ and $\mathcal{B}'$ form a simple pregap that includes $\mathcal{A}, \mathcal{B}$. Clearly, if $\mathcal{A}, \mathcal{B}$ is a gap then $\mathcal{A}', \mathcal{B}'$ is a gap as well. ∎

By the following result, analytic Hausdorff gaps over EU-ideals (if they exist) must be rather different from known analytic Hausdorff gaps (see the proof of [4, Lemma 2]).

THEOREM 2.4: *Assume $\mathcal{Z}_\nu$ is an EU-ideal and $\mathcal{A}, \mathcal{B}$ is an analytic Hausdorff pregap in its quotient. Then every infinite $Y \subseteq \mathbb{N}$ has an infinite subset X such that $\mathcal{A}, \mathcal{B}$ is separated on $\bigcup_{n \in X} I_n$.*

Proof: Assume $\mathcal{A}, \mathcal{B}$ is an analytic Hausdorff gap over $\mathcal{Z}_\nu$. By Theorem 2.1 we may assume $\mathcal{A}, \mathcal{B}$ is a simple gap given by submeasures σ_m, τ_m ($m \in \mathbb{N}$). Since $\mathcal{P}(\mathbb{N})/\operatorname{Fin}$ adds a selective ultrafilter without adding reals, and therefore without splitting gaps, we may assume there exists a selective ultrafilter $\mathcal{U}$ concentrating on Y. Let

$$\mathbb{A} = \left(\prod_n \mathcal{P}(I_n)\right)/\mathcal{U}$$

and define $\nu_{\mathcal{U}}$ on $\mathbb{A}$ as in Lemma 1.2. Identify $D \subseteq \mathbb{N}$ with the element $\langle D \cap I_n : n \in \mathbb{N}\rangle$ of $\prod_n \mathcal{P}(I_n)$, and let

$$\mathcal{A}_{\mathcal{U}} = \{[A]_{\mathcal{U}} : A \in \mathcal{A}\} \qquad \text{and} \qquad \mathcal{B}_{\mathcal{U}} = \{[B]_{\mathcal{U}} : B \in \mathcal{B}\}.$$

These two families form a pregap in $\mathbb{A}/\operatorname{Null}(\nu_{\mathcal{U}})$. By Lemma 1.2, the algebra $\mathbb{A}/\operatorname{Null}(\nu_{\mathcal{U}})$ is a measure algebra and therefore some $[W]_{\mathcal{U}}$ splits the pregap. Let $W_n = W \cap I_n$ and for each k define

$$X_k = \{n | \sigma_n(I_n \setminus W_n) \leq 1/k \text{ and } \tau_n(W_n) \leq 1/k\}.$$

Then $X_k \in \mathcal{U}$ for all k, and since $\mathcal{U}$ is selective we can find $X \in \mathcal{U}$ such that $X \setminus X_k$ is finite for all k. Then X is clearly as required. ∎

3. On quotients without analytic Hausdorff gaps

We prove that if an analytic P-ideal $\mathcal{I}$ is dense and does not have analytic Hausdorff gaps in its quotient, then its restriction to some positive set is a generalized density ideal. This improves the main result of [4] that every dense F_σ P-ideal has such a gap in its quotient since a dense generalized density ideal cannot be F_σ. To this effect we prove a slight strengthening of [4, Lemma 2].

LEMMA 3.1: *Assume $\mathcal{I} = \mathrm{Exh}(\phi)$ is a dense analytic P-ideal and I_i $(i \in \mathbb{N})$ are finite pairwise disjoint sets such that for some $\varepsilon > 0$ and $a > 0$ we have*

(1) $(\forall n)(\forall S \subseteq \bigcup_{i=n}^{\infty} I_i)((\forall i \geq n)\phi(I_i \setminus S) < \varepsilon$
$\Rightarrow (\exists B \subseteq S)(\forall i)\phi(B \cap I_i) < 1/n \wedge \phi(B) > a))$.

Then there is an analytic Hausdorff gap over $\mathcal{I}$.

Proof: By replacing ϕ with ϕ/a we may assume $a = 1$. Recursively find an increasing sequence n_k $(k \in \mathbb{N})$ so that for every k we have (let $J_k = [n_k, n_{k+1})$)

(2) $(\forall S \subseteq \bigcup_{i \in J_k} I_i)(\forall i \in J_k)\phi(I_i \setminus S) < \varepsilon$
$\Rightarrow (\exists B \subseteq S)(\forall i)(\phi(B \cap I_i) < 1/k^2) \wedge \phi(B) > 1)$.

If $n_1, \dots, n_k$ are as required, let T be the family of all pairs (S, p) so that $p > n_k$, $S \subseteq \bigcup_{i=n_k}^{p} I_i$, $\phi(I_i \setminus S) < \varepsilon$ for all $i \in [n_k, p]$, but for every $B \subseteq S$ such that $(\forall i)\phi(B \cap I_i) < 1/k^2$ we have $\phi(B) \leq 1$. Order T by $(S,p) \preceq (U,l)$ if and only if $p \leq l$ and $U \cap I_i = S \cap I_i$ for all $i \leq p$. Then T is a finitely branching tree.

An infinite branch of T would give some S contradicting the assumption (2), since ϕ is lower semicontinuous. By König's lemma,

$$n_{k+1} = \sup\{p + 2 | \exists (S,p) \in T\}$$

is finite and satisfies (2). From this point on we follow the proof of [4, Lemma 2] rather closely.

For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ define submeasures $\alpha_n(A) = |\{j \in J_n : A \cap I_j \neq \emptyset\}|$ and $\beta_n(A) = \sup_{j \in J_n} \phi(A \cap I_j)$, then let

$$\alpha(A) = \sum_{n=1}^{\infty} \frac{\alpha_n(A)}{n} \qquad \text{and} \qquad \beta(A) = \sup_{n \in \mathbb{N}} n \cdot \beta_n(A).$$

Both α and β are lower semicontinuous. We will prove that $\mathcal{A} = \mathrm{Exh}(\alpha)$ and $\mathcal{B} = \mathrm{Exh}(\beta)$ form an analytic Hausdorff gap. Since both are clearly analytic P-ideals, we need only prove that $\mathcal{A}$ and $\mathcal{B}$ are $\mathrm{Exh}(\phi)$-orthogonal and that they are not separated by a single set over $\mathrm{Exh}(\phi)$.

In order to prove $\mathcal{A}$ and $\mathcal{B}$ are $\mathrm{Exh}(\phi)$-orthogonal, note that for $A, B \subseteq \mathbb{N}$ we have

$$\phi(A \cap B) \leq \sum_{n=1}^{\infty} \phi(A \cap B \cap \bigcup\nolimits_{i \in J_n} I_i) \leq \sum_{n=1}^{\infty} \frac{\alpha_n(A)}{n} \cdot (n \cdot \beta_n(B)) \leq \alpha(A) \cdot \beta(B).$$

If $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then $\alpha(A) < \infty$ and $\lim_{m \to \infty} \beta(B \setminus \bigcup_{n=1}^{m} J_n) = 0$, thus by the above $\lim_{l \to \infty} \phi((A \cap B) \setminus [1, l)) = 0$, and $A \cap B \in \mathrm{Exh}(\phi)$, as required.

Assume $\mathcal{A}$ and $\mathcal{B}$ are separated over $\mathrm{Exh}(\phi)$ by $C \subseteq \mathbb{N}$. Then $A \setminus C \in \mathrm{Exh}(\phi)$ and $B \cap C \in \mathrm{Exh}(\phi)$ for all $A \in \mathcal{A}$ and all $B \in \mathcal{B}$. We claim that

(3) $\lim_{n \to \infty} \sup_{m \geq n, j \in J_m} \phi(I_j \setminus C) = 0$.

Otherwise, we may find an infinite $X \subseteq \mathbb{N}$, $\varepsilon > 0$, and a 'choice function' $f \in \prod_{n \in X} J_n$ such that

$$\phi(I_{f(n)} \setminus C) > \varepsilon$$

for all $n \in X$. We may furthermore shrink X so that $\sum_{n \in X} 1/n < \infty$. Let $A = \bigcup_{n \in X} I_{f(n)} \setminus C$; then $\alpha(A) \leq \sum_{n \in X} 1/n < \infty$, thus $A \in \mathcal{A}$. Note that $A \cap C = \emptyset$. However, for $n \in X$ we have $\phi(A \cap \bigcup_{i \in J_n} I_i) \geq \phi(A \cap I_{f(n)}) \geq \varepsilon$, therefore $A \notin \mathrm{Exh}(\phi)$, contradicting the assumption on C.

By (3), for all but finitely many n we have $\sup_{j \in J_n} \phi(I_j \setminus C) < \varepsilon$. By (2), for each such n there is $B_n \subseteq C \cap \bigcup_{i \in J_n} I_i$ such that $\phi(B_n \cap I_i) < 1/n^2$ and $\phi(B_n) \geq 1$. Then $B = \bigcup_{n \in Y} B_n$ satisfies $B \subseteq C$ and $n \cdot \beta_n(B) \leq 1/n$. Therefore $B \in \mathcal{B}$, yet $B \notin \mathrm{Exh}(\phi)$, a contradiction. This completes the proof. ∎

Proposition 3.2: *If $\mathcal{I}$ is an analytic P-ideal whose quotient does not have analytic Hausdorff gaps, then the restriction of $\mathcal{I}$ to some positive set is a generalized density ideal.*

Proof: By [16] fix a lower semicontinuous submeasure ϕ such that $\mathcal{I} = \mathrm{Exh}(\phi)$. Fix a partition of $\mathbb{N}$ into intervals I_i ($i \in \mathbb{N}$) so that $\inf_i \phi(I_i) \geq 1$. The conditions of Lemma 3.1 fail when $a = \varepsilon = 1/m$ for every $m \in \mathbb{N}$. Hence we may assume that for every $m \in \mathbb{N}$ there are $n = f(m)$ and $S \subseteq \bigcup_{i=f(m)}^{\infty} I_i$ such that $(\forall i \geq f(m))\phi(I_i \setminus S) < 1/f(m)$ and if $B \subseteq S$ is such that $\phi(B \cap I_i) < 1/f(m)$ for all i, then $\phi(B) \leq 1/m$. Fix $\delta > 0$ so that $\delta < \inf_i \phi(I_i)$. We may assume $f(m) \geq m/\delta$ for all m. For $k \in \mathbb{N}$ pick $S_k \subseteq \bigcup_{i=f(2^k)}^{\infty} I_i$ so that $\phi(I_i \setminus S_k) < 2^{-k}\delta$ for all $i \geq f(2^k)$ and

$$(\forall B \subseteq S_k)(\forall i)\phi(B \cap I_i) < 1/f(2^k) \Rightarrow \phi(B) < 2^{-k}.$$

Let $S_k' = S_k \cup \bigcup_{i=1}^{f(2^k)-1} I_i$ and $S = \bigcap_{k=1}^{\infty} S_k'$. Then $\phi(I_i \setminus S) < \delta$ for all i, therefore S is $\mathcal{I}$-positive.

We claim that $\{A \subseteq S | A \in \mathcal{I}\} = \{A \subseteq S | \limsup_i \phi(A \cap I_i) = 0\}$, and therefore the restriction of $\mathcal{I}$ to S is a generalized density ideal.

It will suffice to prove that if $\phi(A \cap I_i)$ approaches zero then $A \in \mathcal{I}$. Fix $m \in \mathbb{N}$. Find k such that $\phi(A \cap I_i) < 1/f(2^m)$ for all $i \geq k$. Then $\phi(A \setminus \bigcup_{i=1}^k I_i) < 2^{-m}$, and therefore $A \in \mathrm{Exh}(\phi)$. ∎

4. Concluding remarks

The question whether there are analytic Hausdorff gaps over $\mathcal{Z}_0$ remains open. We record two of its equivalent reformulations. For terminology see [3].

PROPOSITION 1: *Let $\mathcal{I}$ be an analytic ideal. The following are equivalent.*

(a) *There are analytic Hausdorff gaps over $\mathcal{I}$.*
(b) *Every Baire monomorphism of the quotient over $\mathcal{I}$ into an analytic quotient preserves all Hausdorff gaps.*
(c) *Assuming OCA and MA, every monomorphism of the quotient over $\mathcal{I}$ into an analytic quotient preserves all Hausdorff gaps.*

Proof: Each one of (b) and (c) is equivalent to (a) by [3, Proposition 5.9.1 and Proposition 5.9.4]. These equivalences are also implicit in [20]. ∎

Let us repeat [4, Question 9] (see [4, Lemma 2] for a partial answer).

QUESTION 4.1: *Assume a dense analytic P-ideal is equal to* $\mathrm{Exh}(\phi)$ *for a lower semicontinuous submeasure satisfying $\phi(\mathbb{N}) = \infty$. Is there an analytic Hausdorff gap in its quotient?*

A simple argument using the ideas from [5, Proposition 3.3 (1) and (2)] shows that if $\mathcal{Z}_0 = \mathrm{Exh}(\phi)$ for a lower semicontinuous ϕ then $\phi(\mathbb{N}) < \infty$.

Theorem 1.3 implies that $\mathcal{P}(\mathbb{N})/\mathcal{Z}_0$ is a proper forcing notion. The question of properness of quotients $\mathcal{P}(\mathbb{N})/\mathcal{I}$ as forcing notions, initiated by Balcar, has recently attracted considerable attention. Balcar, Hernández Hernández and Hrušák ([1]) proved that $\mathcal{P}(\mathbb{Q})/\mathrm{NWD}(\mathbb{Q})$ is proper and adds only Cohen reals. (Here $\mathrm{NWD}(\mathbb{Q})$ stands for the $F_{\sigma\delta}$ ideal of all nowhere dense subsets of the rationals.) Motivated by [5], Steprāns ([17]) has defined a family of $2^{\aleph_0}$ coanalytic ideals whose quotients are pairwise nonequivalent proper forcing notions, each one being an iteration of a Sacks-like forcing and $\mathcal{P}(\mathbb{N})/\mathrm{Fin}$. Hrušák and Zapletal ([12]) proved theorems relating forcings $\mathcal{P}(\mathbb{N})/\mathcal{I}$ with more familiar forcings of the form Borel/J for a σ-ideal J in a spirit similar to Theorem 1.3. They have also constructed an analytic P-ideal $\mathcal{I}$ such that the forcing $\mathcal{P}(\mathbb{N})/\mathcal{I}$ collapses $\aleph_1$, answering a question from an earlier version of this paper.

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